Last time: we encountered the

	of a matrix $A \in \mathbb{R}^{m\times n}$	of a function $f: W \to V$
rank	$\dim \mathrm{Col}(A)$	$\dim \mathrm{Im}(f)$
nullity	$\dim \mathrm{Null}(A)$ $\Vert$ $\dim \mathrm{Ker}(A)$	$\dim \mathrm{Ker}(f)$

The rank-nullity theorem states: rank + nullity $= n = \dim(W)$

We also encountered change of basis for matrices. If

- $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ is given by $f(x) = Bx$

  basis $\underline{w} = \{w_1, \ldots, w_n\}$ basis $\underline{v} = \{v_1, \ldots, v_m\}$.

- $\mathbb{R}^m \xrightarrow{g} \mathbb{R}^m$ is given by $g(x) = P_{\underline{v}}^{-1} x$

  $v_i \rightsquigarrow e_i$ matrix with columns v_i

- $\mathbb{R}^n \xrightarrow{h} \mathbb{R}^n$ is given by $h(x) = P_{\underline{w}}^{-1} x$

  $w_i \rightsquigarrow e_i$ matrix with columns w_i

Then the composition $\Phi : \mathbb{R}^n \xleftarrow{h} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^m$

i.e. $\Phi = g \circ f \circ h^{-1}$

is given by (i.e. $\Phi(x) = Ax$) the matrix A prescribed by the change of basis formula

$$A = P_{\underline{V}}^{-1} \, B \, P_{\underline{W}}$$

$$\Updownarrow$$

$$B = P_{\underline{V}} \, A \, P_{\underline{W}}^{-1}$$

In particular, for given B of rank r, we can arrange it so that

$$B = P \begin{pmatrix} \overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{r} & 0 \\ 0 & 0 \end{pmatrix} Q$$

for suitably chosen $P \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{n \times n}$; this means that $f(x) = Bx$ does the following w.r.t to bases $w_1, \dots, w_n$ of $\mathbb{R}^n$, $v_1, \dots, v_m$ of $\mathbb{R}^m$

$$\begin{aligned} w_1 &\rightsquigarrow v_1 \\ &\vdots \\ w_r &\rightsquigarrow v_r \\ w_{r+1} &\rightsquigarrow 0 \\ &\vdots \\ w_n &\rightsquigarrow 0 \end{aligned}$$

Today: how to find such P and Q? Do both row and column operations. Recall that

- multiply j-th row by λ : $A \rightsquigarrow D_j^{(\lambda)} A$
- swap rows i and j : $A \rightsquigarrow S_{ij} A$
- add $\lambda\cdot$ row j to row i $A \rightsquigarrow T_{ij}^{(\lambda)} A$
- multiply j-th column by λ : $A \rightsquigarrow A D_j^{(\lambda)}$
- swap columns i and j : $A \rightsquigarrow A S_{ij}$
- add $\lambda\cdot$ column i to column j : $A \rightsquigarrow A T_{ij}^{(\lambda)}$

Example: $B = \begin{pmatrix} 5 & 10 & 15 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{swap rows 1 and 2}} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 5 & 10 & 15 \end{pmatrix}$

$B \rightsquigarrow S_{12}B$

$\xrightarrow{\text{add (-5) row 1 to row 2}} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add (-2) col 1 to col 2}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{add (-3) col 1 to col 3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$S_{12}B \rightsquigarrow T_{21}^{(-5)} S_{12}B$ $\quad T_{21}^{(-5)} S_{12}B \rightsquigarrow T_{21}^{(-5)} S_{12}B T_{12}^{(-2)}$ $\quad T_{21}^{(-5)} S_{12}B T_{12}^{(-2)} \rightsquigarrow T_{21}^{(-5)} S_{12}B T_{12}^{(-2)} T_{13}^{(-3)}$

$$\underbrace{T_{21}^{(-5)} S_{12}}\, B\, \overbrace{T_{12}^{(-2)} T_{13}^{(-3)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{12}\, B\, T_{12}^{(-2)}\, T_{13}^{(-3)} = T_{21}^{(5)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B\, T_{12}^{(-2)}\, T_{13}^{(-3)} = S_{12}\, T_{21}^{(5)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B\, T_{12}^{(-2)} = S_{12}\, T_{21}^{(5)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_{13}^{(3)}$$

$$B = \underbrace{S_{12}\, T_{21}^{(5)}}_{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{T_{13}^{(3)}\, T_{12}^{(2)}}_{Q}$$

Today: We will consider the case when $V = W$ i.e. $v_i = w_i$

we are changing both domain and codomain to the same basis

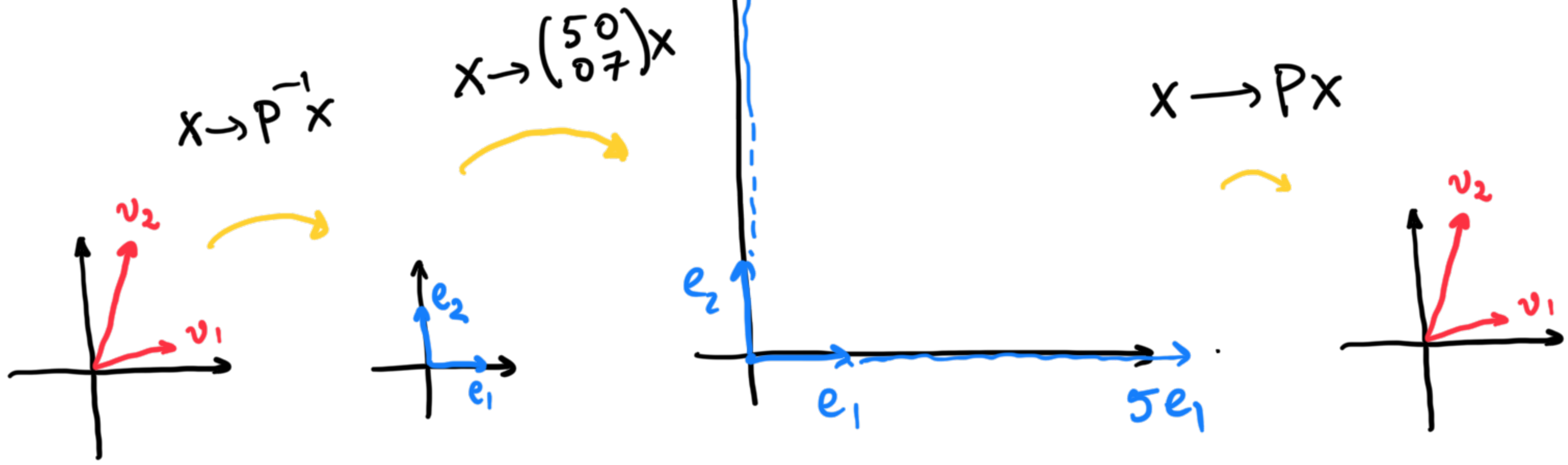
$$\Updownarrow$$

$$B = P A P^{-1} \quad \text{(conjugation)}$$

Recall the example from last time: $B = \frac{1}{5}\begin{pmatrix} 23 & 4 \\ -6 & 37 \end{pmatrix}$

dilates by 5 in the $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

dilates by 7 in the $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$



where $P = P_{\underline{e} \leftarrow \underline{v}} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

Upshot: $$B = P \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} P^{-1}$$

eigenvectors eigenvalues

DEF 17.1: A square matrix B is called **diagonalizable** if it is conjugate to a diagonal matrix, i.e.

$$B = P \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} P^{-1}$$

some invertible matrix some numbers

Application: suppose B is a big square matrix; how to calculate B^N for big N?

assume $B = PDP^{-1}$ where D is diagonal

$$\Downarrow$$

$$B^2 = P D \underbrace{P^{-1} P}_{I} D P^{-1} = P D^2 P^{-1}$$

$$\Downarrow$$

$$B^3 = P D^2 \underbrace{P^{-1} P}_{I} D P^{-1} = P D^3 P^{-1}$$

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$$B^N = P D^N P^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \rightsquigarrow D^N = \begin{pmatrix} \lambda_1^N & & 0 \\ & \ddots & \\ 0 & & \lambda_n^N \end{pmatrix}$$

THM 17.2: Almost any square matrix is diagonalizable.

Non-Ex: $\mathbb{P}_d \xrightarrow{\text{derivative}} \mathbb{P}_d \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & & & \end{pmatrix}$

$(1, x, x^2, \dots, x^d) \leftrightarrow (e_1, e_2, e_3, \dots, e_{d+1})$

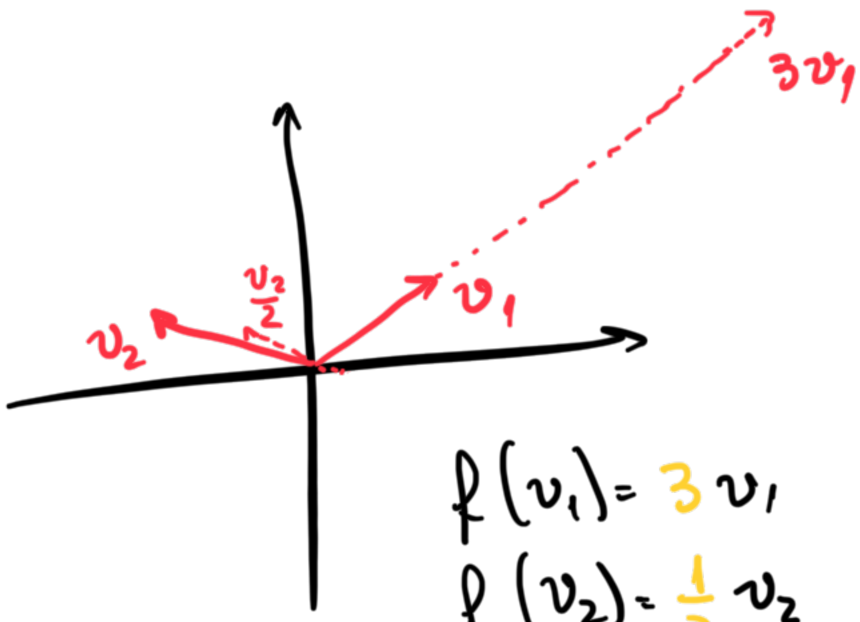
not-diagonalizable

Principle: a matrix $B \in \mathbb{R}^{n \times n}$ is diagonalizable $\Longleftrightarrow$ $\exists$ basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ s.t. B dilates v_i by a factor of λ_i, $\forall i \in \{1, \dots, n\}$

Are geometric linear functions $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$, $f(x) = Bx$ diagonalizable?

- dilations: $B = P \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} P^{-1}$

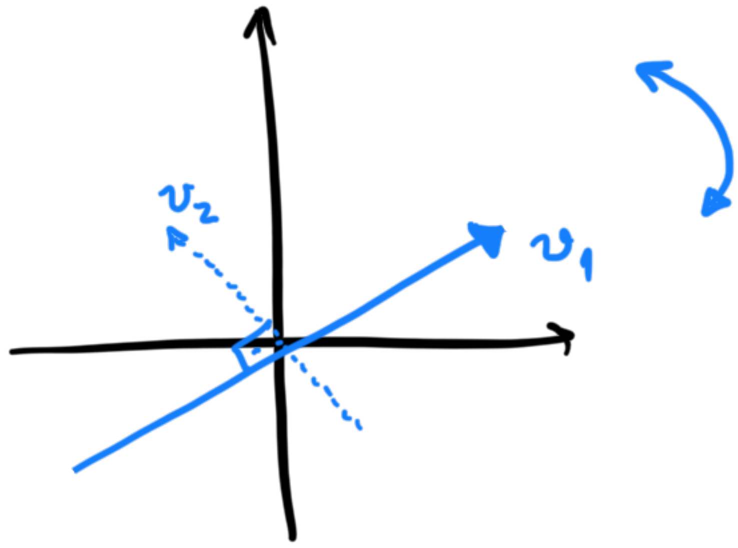
  where $P = (v_1 \mid v_2)$



$f(v_1) = 3\, v_1$
$f(v_2) = \frac{1}{2} v_2$

- reflection: $B = P \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}$

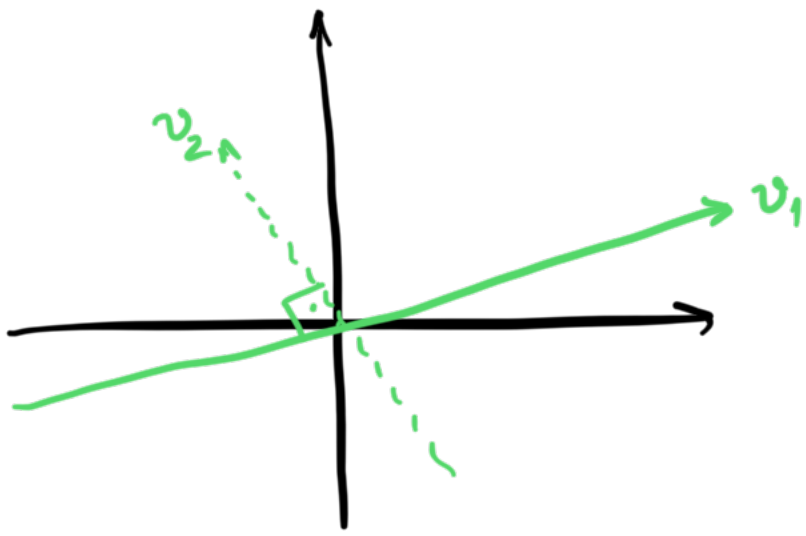
  where $P = (v_1 \mid v_2)$



$f(v_1) = 1 \cdot v_1$
$f(v_2) = (-1)\, v_2$

- projection $B = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$

  where $P = (v_1 \mid v_2)$



$f(v_1) = 1 \cdot v_1$
$f(v_2) = 0 \cdot v_2$

- shearings are not diagonalizable

$$\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \neq P \cdot \text{diagonal} \cdot P^{-1} \quad \text{unless } \mu = 0$$

- rotations by any angle are diagonalizable

$\Updownarrow$

$\left.\begin{array}{l} \text{dilates a vector } v_1 \text{ by a factor of } \lambda_1 \\ \text{dilates a vector } v_2 \text{ by a factor of } \lambda_2 \end{array}\right\}$ possible if $\lambda_1, \lambda_2, v_1, v_2$ are complex

- what is -7 if not a funny symbol s.t.

$$(-7) + 7 = 0 \quad ?$$

- what is $\frac{1}{3}$ if not a funny symbol s.t.

$$\frac{1}{3} \cdot 3 = 1 \quad ?$$

- what is $\sqrt{7}$ if not a funny symbol s.t.

$$(\sqrt{7})^2 = 7 \quad ?$$

- what is i if not a funny symbol s.t.

$$i^2 = -1 \quad ?$$

DEF 17.3 a **complex number** is an expression $z = a + bi = a + ib$ where i is a symbol s.t. $i^2 = -1$ and $a, b \in \mathbb{R}$

Let $\mathbb{C}$ denote the set of complex numbers

- real part $\quad \operatorname{Re} z = a$
- imaginary part $\quad \operatorname{Im} z = b$

$\in \mathbb{R}$

- addition : $(3+5i) + \left(-\frac{2}{3} + \frac{7}{2} i\right) = \frac{7}{3} + \frac{17}{2} i$

$$(3+5i) - \left(-\frac{2}{3} + \frac{7}{2} i\right) = \frac{11}{3} + \frac{3}{2} i$$

- multiplication $(3+i)(-1+2i) = (3+i)(-1) + (3+i)(2i)$

$$= -3 - i + 6i + 2\underbrace{i^2}_{=-1}$$

$$= -3 - i + 6i - 2 = -5 + 5i$$

- division $\dfrac{3+i}{-1+2i} = \dfrac{(3+i)(-1-2i)}{(-1+2i)(-1-2i)} = \dfrac{-3 - i - 6i - 2\overbrace{i^2}^{=-1}}{1 - 2i + 2i - 4\underbrace{i^2}_{=-1}} = \dfrac{-1-7i}{5} = -\frac{1}{5} - \frac{7}{5} i$

DEF 17.4 : $\forall z = a + bi \in \mathbb{C}$, its

DEF 17.4: $\forall\ z = a+bi \in \mathbb{C}$,

- conjugate is $\overline{z} = a - bi \in \mathbb{C}$

$$\left(\overline{2+3i} = 2-3i\right)$$

- absolute value is $|z| = \sqrt{a^2+b^2} \in \mathbb{R}_{\geq 0}$

PROP 17.5: $z \cdot \overline{z} = |z|^2$

$$(a+bi)\cdot(a-bi) = a^2 + bai - abi - b^2 i^2 = a^2 + b^2$$

$$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{z \cdot \overline{w}}{|w|^2} = \frac{\text{complex}}{\text{real}} = \text{complex}$$

THM 17.6:

never divide by 0

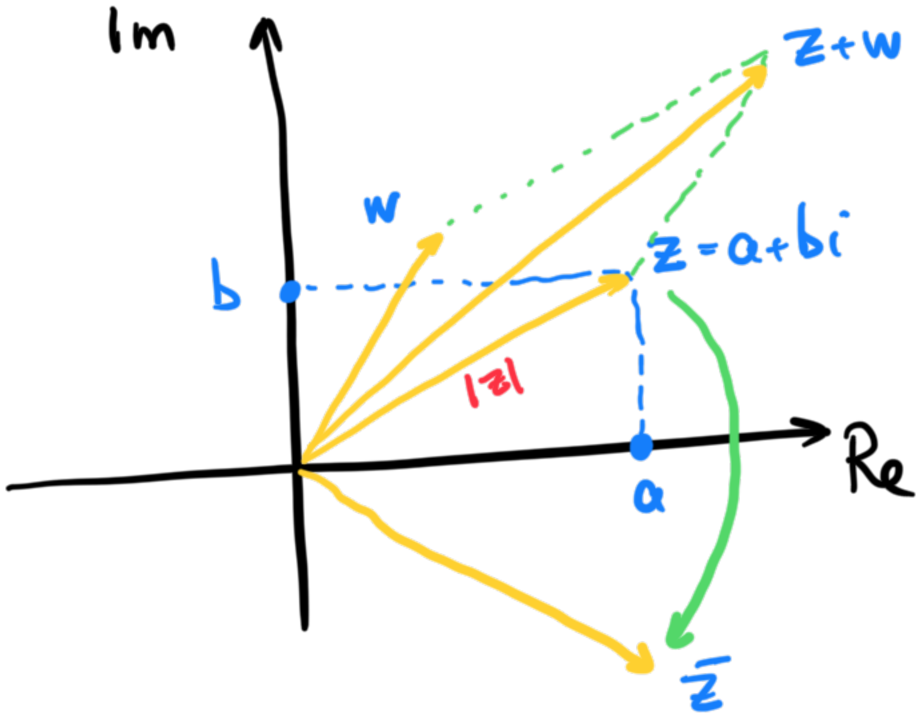
$$\overline{z \pm w} = \overline{z} \pm \overline{w}$$

$$\overline{zw} = \overline{z}\,\overline{w} \quad , \quad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

$$|zw| = |z||w| \quad , \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$\overline{\left(\overline{z}\right)} = z$$

Geometric interpretation of complex numbers



$z = a+bi$ ⇝

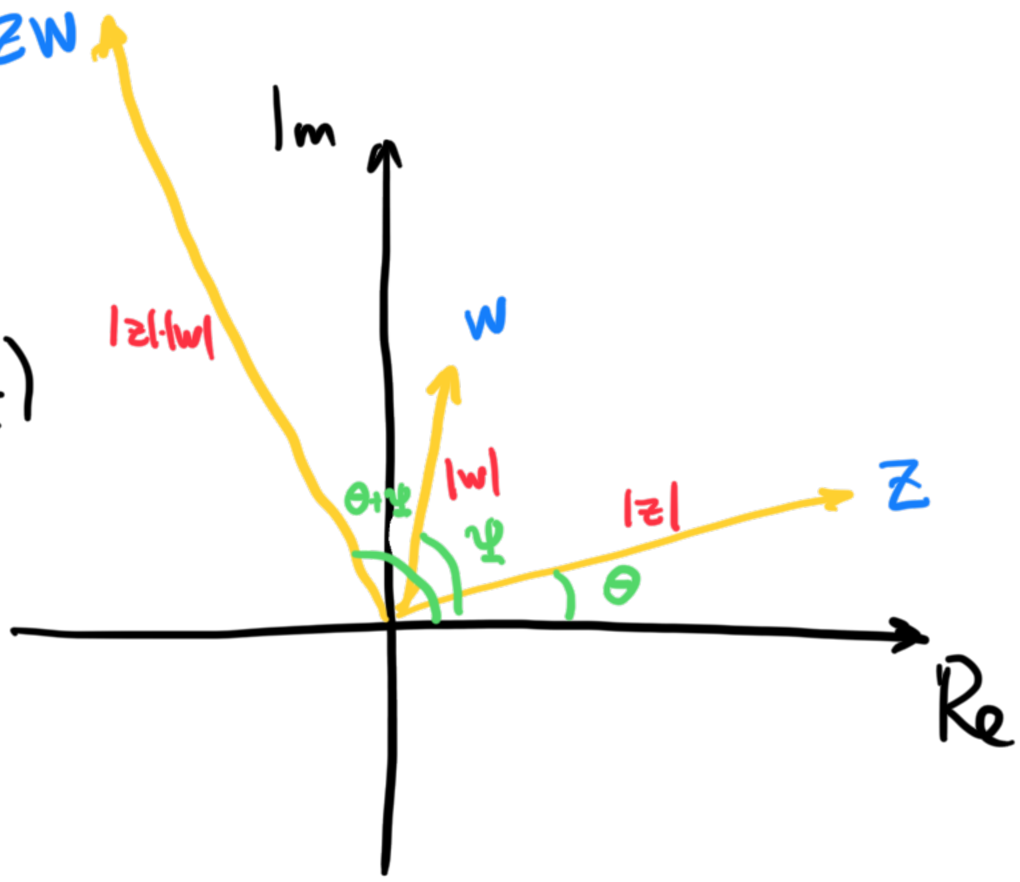
⇓

polar coordinates (r, θ)

$a = r \cdot \cos\theta$
$b = r \cdot \sin\theta$ $\Rightarrow \theta = \arctan\left(\frac{b}{a}\right)$

⇓

$z = r\left(\cos\theta + i \sin\theta\right)$



where $r = |z| \in \mathbb{R}_{\geq 0}$
$\theta = \arg(z) \in [0, 2\pi)$ is called argument of z

PROP 17.7: $|zw| = |z| \cdot |w|$
$\arg(zw) = \arg(z) + \arg(w)$ modulo 2π

Example: raising complex numbers to powers

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$\Downarrow$

z^3 has absolute value $1^3 = 1$

and argument $3 \cdot \frac{\pi}{3} = \pi$

$\Downarrow$

$$z^3 = -1$$

$\cdot\ |z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\cdot\ \arg(z) = \arctan(\sqrt{3}) = \frac{\pi}{3}$